

SURFACE WAVES INDUCED BY EXTERNAL PERIODIC PRESSURE IN A FLUID WITH AN UNEVEN BOTTOM

I. V. Sturova

UDC 532.59

The behavior of waves generated by periodic pressure on the free surface is considered within the linear shallow-water theory. The fluid depth is a piecewise-constant function, which implies the presence of a finite-size bottom trench or elevation. For an arbitrary shape of bottom unevenness, the solution of the problem reduces to a system of integral boundary equations. Manifestation of wave-guiding properties of bottom unevenness is illustrated by an example of an extended rectangular elevation.

Key words: linear shallow-water theory, uneven bottom, trapped waves.

Introduction. There are many investigations dealing with propagation of surface waves in a fluid with an uneven bottom (see, e.g., [1]). The influence of variable topography of the bottom on generation of forced surface waves, however, has not been adequately studied [2].

The present paper describes the solution of the problem of generation of surface waves by external periodic pressure. It is assumed that the bottom is even except for a certain finite-size region with the fluid depth varied as a step function. The problem is solved under the assumptions of the linear shallow-water theory. The method of the solution proposed can be used for an arbitrary planform of the bottom trench or elevation, but the main attention is paid to the bottom unevenness in the form of an extended rectangle because an elevation in the form of an infinitely extended underwater ridge possesses wave-guiding properties. It seems of interest to consider the influence of the finite size of the bottom unevenness on manifestation of wave-guiding properties.

Formulation of the Problem. Let the surface of a layer of an inviscid incompressible fluid be subjected to a pressure periodic in time with a frequency ω of the form

$$p(x, y, t) = P(x, y) \exp(-i\omega t), \quad (1)$$

where x and y are the horizontal coordinates and t is the time. Let H_1 be the depth of the fluid inside a domain Ω_1 bounded by the contour S and H_2 be the depth of the fluid outside this domain. Assumptions of a potential fluid flow and low amplitudes of generated surface waves are used. The frequency of external loading is such that the lengths of generated surface waves are significantly greater than the depth of the fluid. This allows us to use the simplest formulation of the problem: the linear shallow-water theory. The domain occupied by the fluid is divided into two subdomains: the bottom unevenness Ω_1 and the remaining fluid Ω_2 .

Assuming that the fluid oscillations are steady in time, we seek the velocity potentials $\varphi_j(x, y, t)$ that describe the motion of the fluid in the domains Ω_j ($j = 1, 2$):

$$\varphi_j(x, y, t) = \Phi_j(x, y) \exp(-i\omega t).$$

The elevation of the free surface $\eta_j(x, y, t) = \zeta_j(x, y) \exp(-i\omega t)$ is determined by the relation

$$\zeta_j = -\frac{iH_j}{\omega} \Delta\Phi_j.$$

We confine ourselves to considering the case where the localized region of external pressure is located inside the domain Ω_1 .

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; sturova@hydro.nsc.ru. Translated from *Prikladnaya Mekhanika i Tekhnicheskaya Fizika*, Vol. 46, No. 1, pp. 70–77, January–February, 2005. Original article submitted March 2, 2004.

According to the linear shallow-water theory, $\Phi_j(x, y)$ ($j = 1, 2$) is found by solving the system of equations

$$\Delta\Phi_1 + k_1^2\Phi_1 = -i\omega P(x, y)/(g\rho H_1) \quad (x, y \in \Omega_1), \quad k_1 = \omega/\sqrt{gH_1}; \quad (2)$$

$$\Delta\Phi_2 + k_2^2\Phi_2 = 0 \quad (x, y \in \Omega_2), \quad k_2 = \omega/\sqrt{gH_2} \quad (3)$$

with matching conditions at the boundary S

$$\Phi_1 = \Phi_2, \quad \gamma \frac{\partial\Phi_1}{\partial n} = \frac{\partial\Phi_2}{\partial n} \quad (x, y \in S), \quad \gamma = \frac{H_1}{H_2} \quad (4)$$

indicating continuity of pressure and mass flux. In Eqs. (2)–(4), $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, ρ is the fluid density, g is the acceleration of gravity, and \mathbf{n} is the external normal to the contour S . The far field should obey the condition of radiation

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial}{\partial r} - ik_2 \right) \Phi_2 = 0, \quad r = \sqrt{x^2 + y^2}, \quad (5)$$

which means that the surface waves are diverging as $r \rightarrow \infty$.

Method of the Solution. We seek the solution of Eq. (2) in the form

$$\Phi_1(x, y) = \Psi(x, y) + \Phi_0(x, y), \quad (6)$$

where the function $\Psi(x, y)$ is to be determined and $\Phi_0(x, y)$ is the solution of the problem of the action of periodic pressure on the surface of a fluid with a constant depth H_1 .

For simplicity, we assume that the pressure $P(x, y)$ in (1) depends only on the quantity $R = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, where x_0 and y_0 are the coordinates of the epicenter of the external pressure region, i.e.,

$$P(x, y) = ag\rho f(R). \quad (7)$$

Here a is a factor with a dimensionality of length and the function $f(R)$ is dimensionless.

The function $\Phi_0(x, y)$ is determined from the equation

$$\Delta\Phi_0 + k_1^2\Phi_0 = -i\omega f(R)/H_1$$

with a radiation condition similar to (5). The solution of this problem is found by integral Fourier transforms and has the form (see, e.g., [3])

$$\Phi_0(R) = \frac{a\omega}{2\rho} \left[\frac{i}{\pi} \text{v.p.} \int_0^\infty \frac{k\tilde{f}(k)J_0(kR)dk}{gk^2H_1 - \omega^2} - \frac{\tilde{f}(k_1)J_0(k_1R)}{2gH_1} \right], \quad (8)$$

where

$$\tilde{f}(k) = 2\pi \int_0^\infty Rf(R)J_0(kR)dR$$

is the Fourier transform of the function $f(R)$, v.p. indicates the integral in the sense of the principal value, and $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind. The second term in square brackets in equality (8) is necessary to satisfy the radiation condition because the integrand always has a simple pole at the point $k = k_1$.

The function $\Psi(x, y)$ satisfies the equation

$$\Delta\Psi + k_1^2\Psi = 0 \quad (x, y \in \Omega_1). \quad (9)$$

To determine the functions $\Psi(x, y)$ and $\Phi_2(x, y)$, we use the method of integral equations previously applied in [4] to study diffraction of surface waves on a rectangular pit.

Equations (3) and (9) are the Helmholtz equations. The corresponding Green function $G(\mathbf{r}, \mathbf{r}_1; k)$ that satisfies, in the general case, the equation

$$\Delta G + k^2G = 2\pi\delta(\mathbf{r} - \mathbf{r}_1)$$

and the radiation condition in the far field has the form

$$G(\mathbf{r}, \mathbf{r}_1; k) = -(i\pi/2)H_0^{(1)}(k\chi), \quad (10)$$

where $\mathbf{r} = (x, y)$, $\mathbf{r}_1 = (x_1, y_1)$, $\chi^2 = (x - x_1)^2 + (y - y_1)^2$, $\delta(\cdot)$ is the Dirac delta function, and $H_0^{(1)}(\cdot)$ is the zeroth-order Hankel function of the first kind.

Using the Green theorem in the domain Ω_1 , we obtain

$$\varepsilon_1 \Psi(\mathbf{r}) + \frac{1}{\pi} \int_S \left[\Psi(\mathbf{r}_1) \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_1; k_1) - G(\mathbf{r}, \mathbf{r}_1; k_1) \frac{\partial \Psi}{\partial n}(\mathbf{r}_1) \right] ds = 0 \quad (x, y \in \Omega_1), \quad (11)$$

where $\varepsilon_1 = 2$ if the point \mathbf{r} is inside S , $\varepsilon_1 = 1$ if \mathbf{r} belongs to the smooth segment of S , and $\varepsilon_1 = \alpha/\pi$ if \mathbf{r} is a corner point; α is the solid angle of observation of the contour S from the point \mathbf{r} . A similar integral relation is valid in the domain Ω_2 :

$$\varepsilon_2 \Phi_2(\mathbf{r}) = \frac{1}{\pi} \int_S \left[\Phi_2(\mathbf{r}_1) \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_1; k_2) - G(\mathbf{r}, \mathbf{r}_1; k_2) \frac{\partial \Phi_2}{\partial n}(\mathbf{r}_1) \right] ds \quad (x, y \in \Omega_2). \quad (12)$$

In this case, we have $\varepsilon_2 = 2$ if the point \mathbf{r} is outside S , $\varepsilon_2 = 1$ if \mathbf{r} is on the smooth segment of S , and $\varepsilon_2 = 2 - \alpha/\pi$ if \mathbf{r} is a corner point of S .

To determine the potentials $\Psi(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ inside the domains Ω_1 and Ω_2 , respectively, we need to calculate the values of $\Psi(\mathbf{r})$ and $\partial\Psi(\mathbf{r})/\partial n$ on the contour S .

Using the points \mathbf{r} on the boundary S , we obtain a system of two integral equations; one of them is Eq. (11), and the other, according to (4), (6), and (12), has the form

$$\varepsilon_2[\Psi(\mathbf{r}) + \Phi_0(\mathbf{r})] = \frac{1}{\pi} \int_S \left\{ \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_1; k_2) [\Psi(\mathbf{r}_1) + \Phi_0(\mathbf{r}_1)] - \gamma G(\mathbf{r}, \mathbf{r}_1; k_2) \left[\frac{\partial \Psi}{\partial n}(\mathbf{r}_1) + \frac{\partial \Phi_0}{\partial n}(\mathbf{r}_1) \right] \right\} ds \quad (x, y \in S). \quad (13)$$

After determining the values of $\Psi(\mathbf{r})$ and $\partial\Psi(\mathbf{r})/\partial n$ on the contour S , we can calculate various parameters of the flow, including vertical displacements of the free surface

$$\zeta_1 = iH_1(k_1^2 \Psi - \Delta \Phi_0)/\omega \quad (x, y \in \Omega_1), \quad \zeta_2 = i\omega \Phi_2/g \quad (x, y \in \Omega_2).$$

Using an asymptotic representation of the Green function (10) as $r \rightarrow \infty$

$$G(\mathbf{r}, \mathbf{r}_1; k_2) \approx -\sqrt{\pi/(2rk_2)} \exp\{i[k_2(r - x_1 \cos \theta - y_1 \sin \theta) + \pi/4]\},$$

we obtain an approximate expression for the potential of surface waves far from the bottom unevenness

$$\Phi_2(r) \approx \sqrt{1/(8\pi k_2 r)} \exp[i(k_2 r + \pi/4)] H(k_2, \theta) \quad (r \rightarrow \infty),$$

where

$$H(k, \theta) = \int_S \left[\frac{\partial \Phi_2}{\partial n} + ik\Phi_2(n_x \cos \theta + n_y \sin \theta) \right] \exp[-ik(x_1 \cos \theta + y_1 \sin \theta)] ds.$$

Here $\theta = \arctan(y/x)$ and n_x, n_y are the corresponding components of the external normal vector \mathbf{n} to the contour S at the point x_1, y_1 .

The Kochin function $H(k_2, \theta)$ is used to express the amplitudes of surface waves in the far field:

$$|\zeta_2| = \frac{\omega}{g} \sqrt{\frac{1}{8\pi k_2 r}} |H(k_2, \theta)|.$$

Rectangular Bottom Unevenness. The results of the method proposed are demonstrated by an example of a rectangular domain Ω_1 of length L and width B . The origin of the coordinate system is located at the center of the domain Ω_1 . In the numerical solution, the segments of the contour S parallel to the x and y axes are divided into N_x and N_y identical segments, respectively. After that, each segment is divided into three identical parts, two auxiliary internal points are introduced, and four-point cubic shape functions are used on each segment. The total number of points where the unknown functions are determined is $M = 6(N_x + N_y)$. A discrete form of Eqs. (11) and (13) is written for each node. The set of these linear algebraic equations forms the system

$$[A_1][\Psi] + [B_1] \left[\frac{\partial \Psi}{\partial n} \right] = 0, \quad [A_2][\Psi] + [B_2] \left[\frac{\partial \Psi}{\partial n} \right] = [C],$$

where $[A_{1,2}]$, $[B_{1,2}]$, and $[C]$ are quadratic matrices of size M ; $[\Psi]$ and $[\partial\Psi/\partial n]$ are the vectors of nodal values of the corresponding functions on the boundary S .

Wave-Guiding Properties of a Rectangular Ridge. We briefly describe the solution of the problem on determining the characteristics of waves trapped by the bottom unevenness of finite width B and infinite length. Let H_1 be the depth of the fluid in the domain $|x| < B/2$, $|y| < \infty$ and H_2 be the depth of the fluid in the domain $|x| > B/2$, $|y| < \infty$. To determine the wave-guiding properties of this bottom unevenness, we have to find a nontrivial solution of homogeneous equations for the corresponding velocity potentials

$$\frac{\partial^2 \varphi_1}{\partial t^2} = gH_1 \Delta \varphi_1 \quad (|x| < B/2, \quad |y| < \infty); \quad (14)$$

$$\frac{\partial^2 \varphi_2}{\partial t^2} = gH_2 \Delta \varphi_2 \quad (|x| > B/2, \quad |y| < \infty) \quad (15)$$

with matching conditions on the lines $|x| = B/2$ similar to (4).

We seek the solutions of Eqs. (14) and (15) in the form

$$\varphi_j(x, y, t) = \Phi_j(x) \exp[i(\lambda y - \omega t)] \quad (j = 1, 2).$$

To determine the functions $\Phi_j(x)$, we have the system of equations

$$\begin{aligned} \Phi_1'' - \sigma^2 \Phi_1 &= 0 \quad (|x| < B/2), & \sigma^2 &= \lambda^2 - k_1^2, \\ \Phi_2'' - \beta^2 \Phi_2 &= 0 \quad (|x| > B/2), & \beta^2 &= \lambda^2 - k_2^2 \end{aligned}$$

with the matching conditions

$$\Phi_1 = \Phi_2, \quad \gamma \Phi_1' = \Phi_2' \quad (|x| = B/2), \quad (16)$$

where the prime denotes differentiation in terms of x .

The following condition should be satisfied in the far field for waves trapped by the bottom unevenness:

$$\Phi_2 \rightarrow 0 \quad (|x| \rightarrow \infty);$$

according to this condition, the solution for $\Phi_2(x)$ has the form

$$\Phi_2 = \alpha_+ \exp(-\beta x) \quad (x > B/2), \quad \Phi_2 = \alpha_- \exp(\beta x) \quad (x < -B/2), \quad (17)$$

where α_{\pm} are unknown constants. There is no wave motion far from the underwater ridge only in the case of real positive values of β , which means that the condition $\lambda > k_2$ is satisfied.

To study eigenoscillations of the fluid, it is convenient to divide them into symmetric and antisymmetric modes with respect to the line $x = 0$. The velocity potential $\Phi_1(x)$ can be represented in the form $\Phi_1 = c_+ \cosh(\sigma x)$ for symmetric modes and in the form $\Phi_1 = c_- \sinh(\sigma x)$ for antisymmetric modes (c_{\pm} are unknown constants).

Using the matching conditions (16) and the representation of solution (17) for $|x| > B/2$, we can easily show that the characteristics of the trapped waves should satisfy the following conditions:

— for symmetric modes,

$$\tanh(\sigma B/2) = -\beta/(\gamma\sigma); \quad (18)$$

— for antisymmetric modes,

$$\tanh(\sigma B/2) = -\gamma\sigma/\beta. \quad (19)$$

Equations (18) and (19) can have real roots only for $\gamma < 1$ in the frequency range $\lambda\sqrt{gH_1} < \omega < \lambda\sqrt{gH_2}$. Hence, only ridge-type unevenness possesses wave-guiding properties.

Numerical Results. Figure 1 shows the characteristics of trapped modes for an underwater ridge of width $B/H_2 = 12$ for $\gamma = 0.5$ in the dimensionless variables $\bar{\omega} = \omega\sqrt{H_2}/g$ and $\bar{\lambda} = \lambda H_2$. It is seen that the symmetric mode has the shortest length of the trapped wave.

The action of periodic pressure is considered for a rectangular unevenness of size $L/H_2 = 60$ and $B/H_2 = 12$. We use the pressure distribution for which the function $f(R)$ in Eq. (7) has the form

$$f(R) = \exp(-bR^2), \quad \tilde{f}(k) = (\pi/b) \exp(-k^2/(4b)). \quad (20)$$

The numerical calculations are performed for $bH_2^2 = 2$. Though the function $f(R)$ in (20) was determined for $0 < R < \infty$, the pressure spot can be considered as localized for the value of b used, because the value of $f(R)$ with $R > 2H_2$ is less than $5 \cdot 10^{-4}$ and rapidly decays with increasing R .

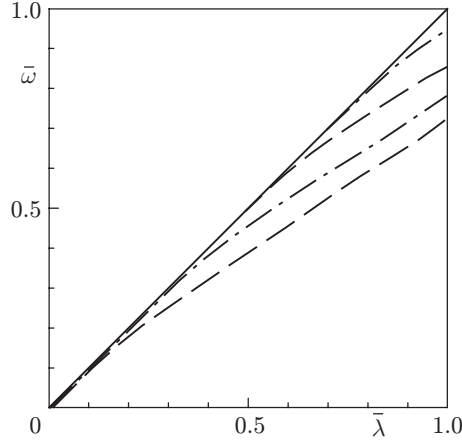


Fig. 1. Frequencies of trapped modes for a rectangular ridge of width $B/H_2 = 12$ with $\gamma = 0.5$: the solid curve corresponds to $\bar{\omega} = \bar{\lambda}$; the dashed and dot-and-dashed curves refer to the symmetric and antisymmetric modes, respectively.

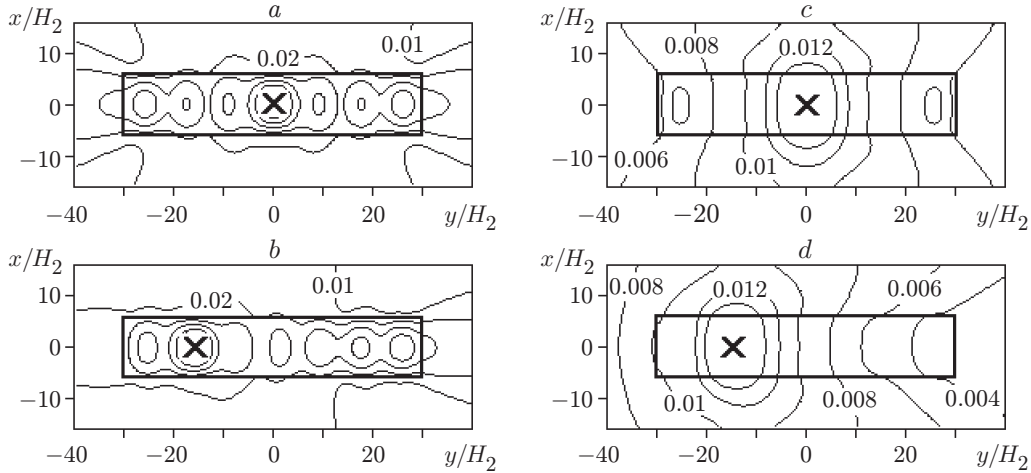


Fig. 2. Isolines of $|\bar{\zeta}|$ for $\bar{\omega} = 0.3$ and $x_0 = 0$: (a) $\gamma = 0.5$ and $y_0 = 0$; (b) $\gamma = 0.5$ and $y_0/H_2 = -15$; (c) $\gamma = 1.5$ and $y_0 = 0$; (d) $\gamma = 1.5$ and $y_0/H_2 = -15$; the cross indicates the epicenter of external pressure.

Figure 2 shows the isolines of amplitudes of free-surface elevations $\bar{\zeta} = \zeta_{1,2}/a$. The domain of the free surface above the bottom unevenness whose projection is indicated in the figure by the internal rectangle and the immediate vicinity of this domain are shown. The isolines of $|\bar{\zeta}|$ corresponding to the levels from 0.01 to 0.05 with a step of 0.01 are plotted in Fig. 2a and b; the levels from 0.004 to 0.014 with a step of 0.002 are plotted in Fig. 2c and d. The epicenter of external pressure corresponds to $x_0 = y_0 = 0$ for Fig. 2a and c and to $x_0 = 0, y_0/H_2 = -15$ for Fig. 2b and d. The values of $|\bar{\zeta}|$ at the pressure epicenter are 1.06, 0.86, 1.03, and 0.83 for Figs. 2a–2d, respectively. The calculations were performed for $N_x = 6$ and $N_y = 30$; a further increase in these parameters involves practically no changes in the results. More detailed information on the behavior of $|\bar{\zeta}|$ on the medial line $x = 0$ outside the main pressure region is shown in Fig. 3. It is seen from Figs. 2 and 3 that, in the presence of a bottom elevation, the preferable direction of surface waves is along this elevation; in the case of a bottom trench, the waves propagate mainly in the transverse direction.

It is convenient to describe the behavior of surface waves far from the bottom unevenness by the scattering diagram. The dependence $D = |\bar{H}|/\sqrt{8\pi\bar{\omega}}$ on the angle θ for $x_0 = 0$ and $y_0/H_2 = -15$, where $\bar{H} = \gamma H(k_2, \theta)/\sqrt{gH_2^3}$, is shown in polar coordinates in Fig. 4. In the presence of an extended bottom elevation (Fig. 4a and c), the surface waves with the highest amplitude propagate in the direction of the long side of the bottom unevenness, i.e., along the y axis, even in the far field.

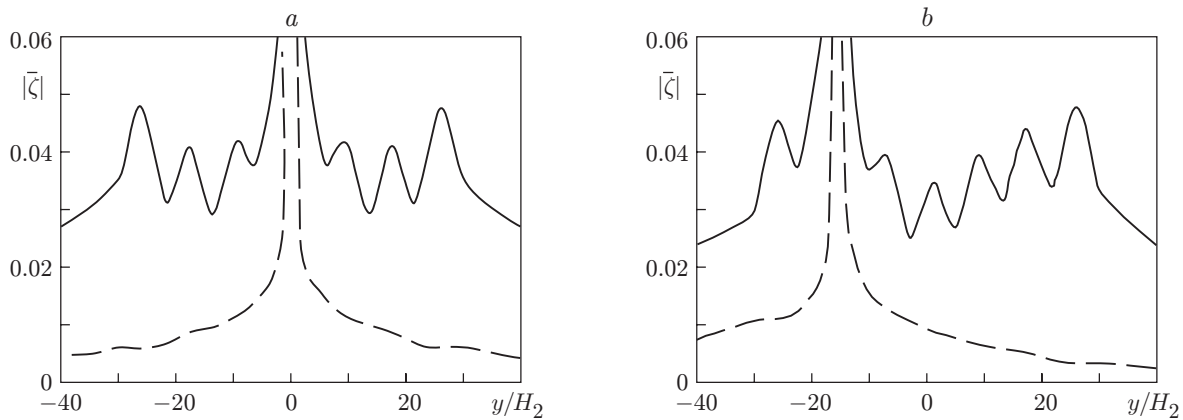


Fig. 3. Distribution of $|\bar{\zeta}|$ along the line $x = 0$ for $\bar{\omega} = 0.3$ and $x_0 = 0$: (a) $y_0 = 0$; (b) $y_0/H_2 = -15$; the solid and dashed curves refer to $\gamma = 0.5$ and $\gamma = 1.5$, respectively.

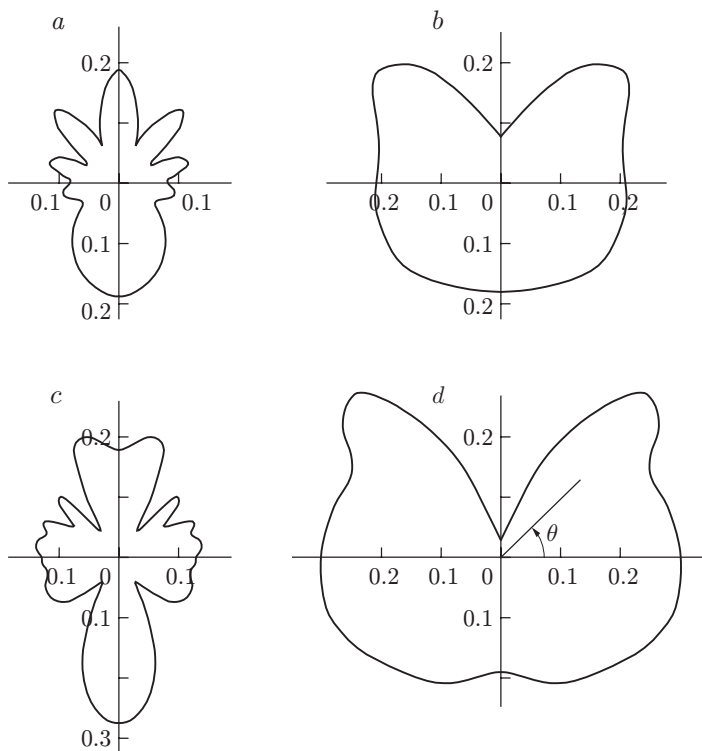


Fig. 4. Scattering diagram $D(\theta)$ for $x_0 = 0$ and $y_0/H_2 = -15$: (a) $\bar{\omega} = 0.3$ and $\gamma = 0.5$; (b) $\bar{\omega} = 0.3$ and $\gamma = 1.5$; (c) $\bar{\omega} = 0.5$ and $\gamma = 0.5$; (d) $\bar{\omega} = 0.5$ and $\gamma = 1.5$.

The results presented show that the method proposed is an effective tool for studying the behavior of long surface waves generated by periodic disturbances in a fluid with an uneven bottom of a particular type. An extended bottom elevation possesses wave-guiding properties, as an infinitely extended ridge does.

This work was partly supported by the Grant of the President of the Russian Federation for Leading Scientific Schools (Grant No. NSh-902.2003.1).

REFERENCES

1. P. H. LeBlond and L. A. Mysak, *Waves in the Ocean*, Elsevier Scientific Publishing Company, Amsterdam (1978).
2. A. A. Korobkin and I. V. Sturova, "Generation of surface and internal waves in a variable-depth fluid," in: *Methods of Hydrophysical Research. Turbulence and Microstructure* (collected scientific papers) [in Russian], Inst. Appl. Phys., Acad. of Sci. of the USSR, Gor'kii (1990), pp. 161–179.
3. L. V. Cherkosov, *Surface and Internal Waves* [in Russian], Naukova Dumka, Kiev (1973).
4. A. N. Williams, "Diffraction of long waves by rectangular pit," *J. Waterway, Port. Coastal, and Ocean Eng.*, **116**, No. 4, 459–469 (1990).